Project 3 – A Discontinuous Galerkin program DG(P1) for solving 1D nonlinear Advection-Diffusion equations

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ABSTRACT
The aim of the project is to augment the understanding of the flux calculation for diffusion problems. In the project, a Discontinuous Galerkin DG(P1) solver is used along with the Taylor Nodal Basis in order to solve the 1-D equation of non-linear advection diffusion. The diffusive fluxes have been calculated using the Direct Discontinuous Galerkin (DDG) Formulation and the advective linear and non-linear fluxed have been formulated using the Upwinding method. The results for the two test cases of (a) Heat Equation and (b) Non-linear Advection Diffusion equation match the exact solution and hence the DDG estimates a very good numerical solution as is even shown by the convergence study for the test case (a).

Keywords: *CFD, Direct Discontinuous Galerkin Method, DDG, Taylor Basis, Advection, Diffusion*

1. Introduction

The Advection and Diffusion equations are the most general form of equations for a physical problem. An advection equation is like any other convection phenomena in the physical world. It is basically a wave-like propagation of any quantity from one place to the other. Advection can represent transport of a fundamental quantity like mass, density or temperature etc. The diffusion phenomena is also one of most widely studied phenomena because of its universal presence. Since there is nothing like an ideal situation in physics hence, the wave that propagates through any viscous medium undergoes a decrease in amplitude of propagation due to the loss of quantity while interacting with the medium of propagation.

Due to the high cost of the experimental setups and equipment, the numerical study of the daily fluids phenomena have taken a toll. Also, due to the availability of high computing devices that are being developed every day, allow for the increasing complexity and exactness of the numerical solution to reach an accurate solution. Computational Fluid Dynamics (CFD) is increasingly becoming popular for the robust analysis in every field of science like aerodynamics, materials testing, weather forecasting etc. Since the amount of math involved is high, there is an increasing need for High Performance Computing to be set up in order to run codes regularly and efficiently.

Solving a fluid dynamics problem requires a set of equations to define the physics

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of the problem. The equations are then applied to the domain in which one needs to find the solution to. The domain now, is discretized into a finite number of cells which are solved for the equations, one at a time. The final solution over the period of time defined is computed.

The discretization methods fall into a number of categories like, (1) Finite Difference methods (2) Finite Volume methods (2) Galerkin methods (continuous or discontinuous); which are used based on the ones that best define the problem and result in a higher accuracy at a lower computational cost.

In the following project, the scalar advection-diffusion equations are solved over a one-dimensional domain and the initial conditions, exact solution and the numerical solution are presented along with the convergence study to analyse the accuracy of the present schemes used.

2. Problem Definition and Solution Methodology

2.1. Description of the Problem

The scalar non-linear advection-diffusion is defined as follows:

$$\partial u/\partial t + \partial(au + b\, u^2/2)/\partial x = c\, (\partial^2\, u)/(\partial x^2\,) \tag{1}$$

Where a,b,c are constants.

Based on the equation (1), there will be two different cases that will be solved.

Case 1: The Heat Equation (a=0, b=0, c=1)

$$\partial u/(\partial t) = c(\partial^2 u)/(\partial x^2) \tag{2}$$

The initial condition being,

$$u(x, 0) = \sin(x);\ x \in [0, 2\pi] \tag{3}$$

Here, we evaluate the final solution at time t=2 seconds

In order to analyse the numerical solution, we conduct a convergence study of 4, 8, 16 and 32 grid cells.

The exact solution of this problem is given by,

$$u(x, t) = e^{(-t)}\ \sin(x) \tag{4}$$

Case2: Non-linear Advection-Diffusion Equation (a=1, b=1, c=1.5 $\times$ 10^{-5})

$$\partial u/\partial t + \partial(au + b\, u^2/2)/\partial x = c\, (\partial^2\, u)/(\partial x^2) \tag{5}$$

The initial condition being

$$u(x, 0) = u_0\, e^{(-\beta x)^{10}} \sin(2\pi x)\ \ ;\ x \in [-10, 40] \tag{6}$$

Where $u_0 = 7.96 \times 10^{-3}$ *and* $\beta = 0.179$

Here, we evaluate the final solution at time t= 30 seconds

The number of cells into consideration here is 4000.

The initial conditions as well as the numerical solutions are plotted in the subsequent section in order to give a broader view of the problem at hand.

2.2. Solution Methodology

The discontinuous Galerkin (DG) method is a class of finite element methods first introduced by Reed and Hill in 1973. A Galerkin finite element method has the characteristic of having the same function space for both the numerical solution and test functions. DG methods are named for their piecewise discontinuous function space, usually chosen to be polynomials, for both the numerical solution and test functions. These robust and accurate methods have quickly attracted the interest of the scientific community.

Using DG methods for diffusion problems have been considered since a long time. This has been a challenging task because of the difficulty in properly defining the numerical solution derivative at cell interfaces. Because the numerical solution is allowed to be discontinuous across cell interfaces, appropriate numerical fluxes for diffusion terms need to be defined.

Recently, a direct discontinuous Galerkin (DDG) method was developed for solving diffusion equations. The scheme is based on the direct weak formulation of the heat equation, and a general numerical flux formula for the solution derivative was proposed. An optimal (k)th order error estimate in an energy norm was obtained for P(k) polynomial approximations of linear diffusion equation.

We consider a simple case of 1-D diffusion. Consider the heat equation in (2). Partition the domain into a number of computation cells Multiply the heat equation by any smooth function and integrate over the domain and
perform integration by parts to formally obtain,

$$\int_\Omega U_t\ V\ dx - (U_x\ )_{(j+1/2)}\ \ (V)_{(j+1/2)}\ + (U_x\ )_{(j-1/2)}\ \ (V)_{(j-1/2)} = \int_\Omega U_x\ V_x dx \tag{7}$$

Replacing the smooth function V, by a test function and the exact solution, we arrive at the original Direct Discontinuous Galerkin formulation [1] for the diffusion equations which is defined as

$$\int_\Omega u_t v\ dx\ -\ \widehat{u_x} v_{j-1/2}^{j+1/2}\ +\ \int_\Omega u_x v_x\ dx = 0 \tag{8}$$

where, the numerical flux is given by

$$\widehat{u_x} v_{j-1/2}^{j+1/2} = \widehat{(u_x)}_{j+1/2} v_{j+1/2}^{-} - \widehat{(u_x)}_{j-1/2} v_{j-1/2}^{+} \tag{9}$$

This scheme is well defined provided that numerical flux is given. The numerical flux was introduced by taking,

$$\widehat{u_x} = \beta_0\ \frac{[u]}{\triangle x} + \overline{u_x} \tag{10}$$

where we adopt the following notation

$$[u]\ =\ u^+ + u^-\ ;\ \overline{u}\ =\ \frac{u^+ + u^-}{2}$$

[u] is known as the jump operator and the equation also contains $\overline{u_x}$ which is the average of the gradients.

3. Results and Discussion

The domain for the test cases have been discretized as done for the general numerical methods and various types of grid sizes were taken into consideration in order to conduct a convergence study at the end of the test case. The diffusion equation is expected to diffuse without any advection whereas the non-linear advection-diffusion equation is expected to show a more complex behaviour over time due to the presence of a Berger's flux which distorts the behaviour of the wave into a shock gradually over time.

3.1. Heat Equation

As defined in equation (2), the Heat equation is considered in this test case and grid sizes of 4,8,16 and 32 cells were taken into account and the general DDG Solver Code was run and the following plots were obtained at a final time of 2 seconds.

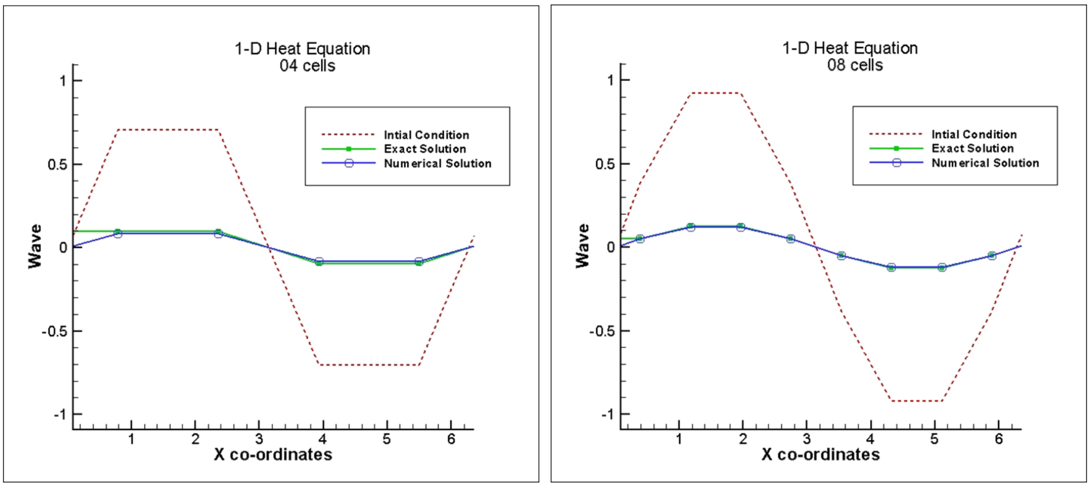

Figure 1. 1-D Heat equation on various grid sizes (a,b)

As we can see from the above figures, as we proceed from figure a to figure d, the smoothness of the grid enhance and hence does the solution. The DDG formulation gives a perfect match with the exact solution as evident from the plots.

The error, on increasing the grid sizes decreases as well. Using the L2 Norm, the error was calculated and the log(error) plot is shown in figure 3. We can see that the logarithm of the error decreases upon the increasing grid size.

3.2. Non-Linear Advection-Diffusion Equation

As described in equation (5), the non-linear advection diffusion was formulated using the first order Upwind method and the diffusive terms were calculated using the DDG formulation as described in the earlier sections. The domain was divided into 4000 cells

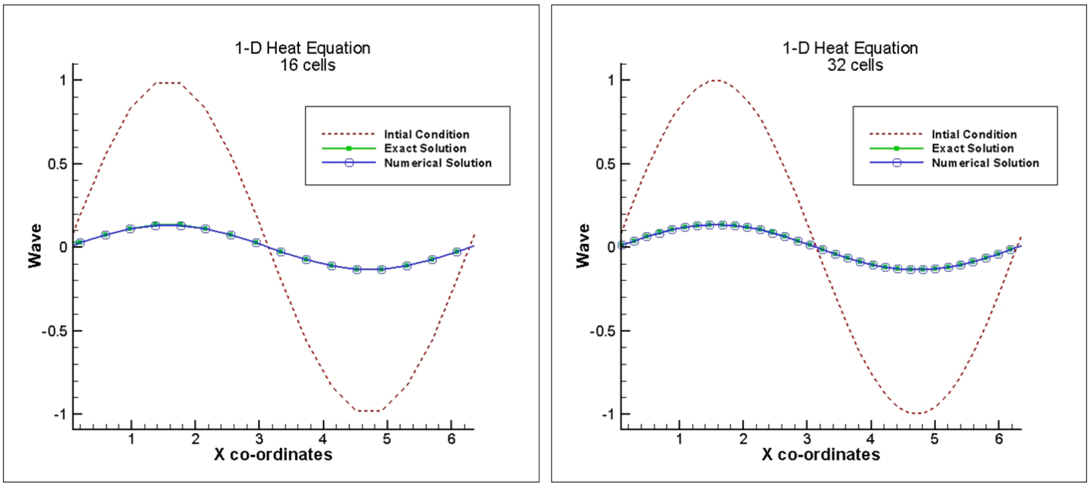

Figure 2. 1-D Heat equation on various grid sizes (c,d)

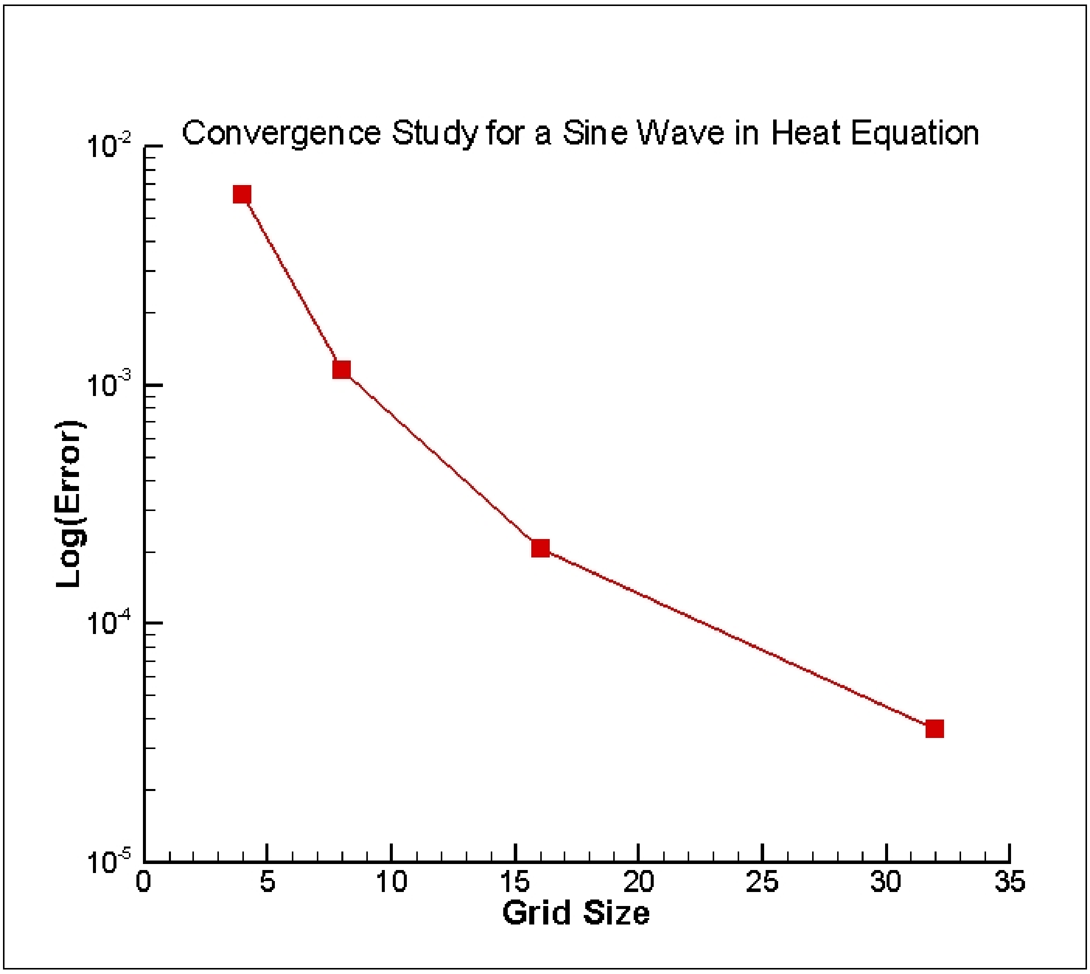

Figure 3. Convergence Study of Sine Wave in Diffusion equation

over a domain from the x-coordinate from -10 to 40. The initial wave and the final wave at time 30 seconds is shown in figure 4.

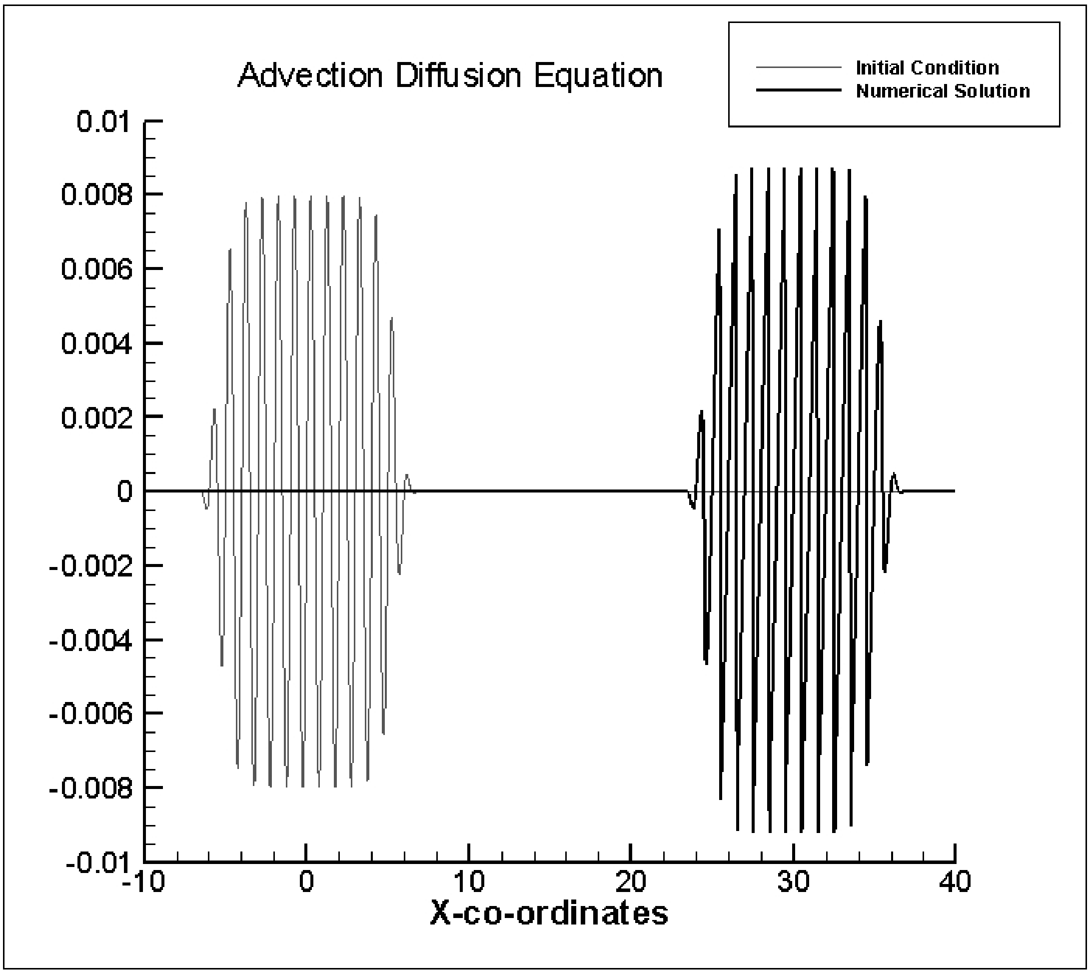

Figure 4. Initial (left) and Final snapshot of the problem.

As can be easily seen in the figure, the initial wave is being advected and diffused over a time period of 30 seconds.

In order to get a closer look at the final condition of the waveform, figure 5 represents the domain from x-coordinate, 29 to 31 which shows hows the initial sine wave from equation (6) is moving toward a shock wave along with diffusion and increased wavy behavior at the crests and troughs.

Comparing now the initial condition in equation (6) and the final plot in figure 5, the difference is easily made out about the distortion of the sine wave into a shock wave over the time.

4. Conclusion

The DDG formulation is an important and useful tool in order to easily describe the flux formulation in the diffusion equations. As seen from the plots and convergence study, the DDG formulation produces near exact solution with DGP1 schemes. A higher order of accuracy can be obtained by including second and third gradients in

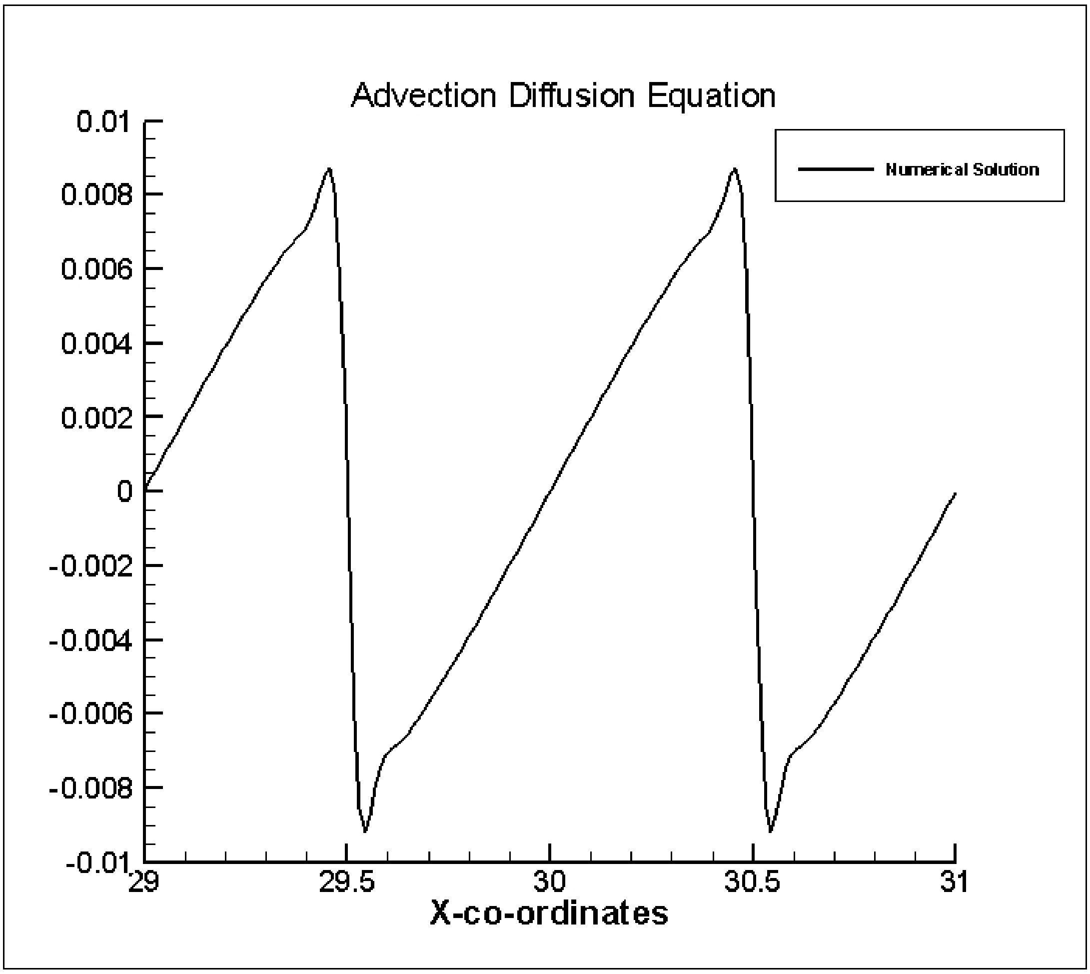

Figure 5. Advection-Diffusion equation at time 30 seconds

the numerical flux term of the DDG formulation which will necessitate a more exact solution with an improved accuracy for more complex diffusion equations.

References

[1] LIU HAILIANG, YAN J. THE DIRECT DISCONTINUOUS GALERKIN (DDG) METHODS FOR DIFFUSION PROBLEMS. SIAM Journal on Numerical Analysis. 2008; 47(1):675–698. Available from: www.jstor.org/stable/25663141.